Int. J. Solids Structures, 1968, Vol. 4, pp. 857 to 873. Pergamon Press. Printed in Great Britain

AN EQUILIBRIUM STRESS FIELD MODEL FOR FINITE ELEMENT SOLUTIONS OF TWO-DIMENSIONAL ELASTOSTATIC PROBLEMS

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Abstract—It is useful to have the dual formulation to the usual compatible element model in order to provide bounds on the exact solution. An element whose stress field satisfies the equilibrium differential equations and which is connected to the adjoining elements in such a manner as to provide continuity of the stress vector across all boundaries furnishes the desired dual.

A suitable equilibrium model is formulated for use in a routine manner in the finite element scheme. Guidelines for the formulation of other elements are given and the causes of troubles which have plagued earlier investigators are isolated.

Numerical results are given which serve to illustrate and validate the ideas presented.

1. INTRODUCTION

THE finite element method in continuum mechanics is most often discussed in terms of displacements. This is done since it is easy to relate trial displacement fields to the behavior of the "nodes" of each element. The required relationships between element nodes, i.e., the compatibility relationships, are also easily written down providing the necessary trial solutions for a Rayleigh–Ritz approach.

For such a formulation (whether finite elements or not) it has long been recognized (see e.g. Synge [1]) that if the trial displacement fields are taken such as to satisfy compatibility throughout the domain, then the solution will provide a bound to the true solution. The opposite bound is provided by ensuring that the trial solutions satisfy equilibrium. This result also has been recognized [1] but has been little employed due to the more difficult task (in details, not conception) of providing trial solutions which satisfy equilibrium and the stress type boundary conditions exactly.

From a "finite element" point of view the equilibrium approach has been studied in depth only by Fraeijs de Veubeke [2–4]. In this earlier work Fraeijs de Veubeke encountered problems in the application of the simple, constant stress field. These problems were of the nature of singularities in the assembled stiffness matrix. These problems were explained by the concept of the "kinematic mode". Special procedures were devised to overcome the problem and applications of a quadrilateral element made up of four constant stress triangles are given in [5]. In a very recent paper [6], Fraeijs de Veubeke and Zienkiewicz make use of the analogy between the Airy stress function and the lateral displacement of a plate to indicate how suitable two dimensional stress equilibrium models may be generated from completely compatible plate bending displacement models. Then, as they

mention, this leads the way to a quadrilateral equilibrium element based on Fraeijs de Veubeke's plate bending quadrilateral. The particular element used as an example herein, which is developed completely and directly from an equilibrium point of view, is almost the dual of the triangular compatible element of Clough and Toucher [7]. The difference is related to the fact that Clough and Toucher left out one of the polynominal terms in the displacement expansion in order to achieve a linear variation of $\partial w/\partial n$ along the exterior boundaries of the element. This duality broadens the interest of the work herein since the fundamental necessity for the element grouping is shown to be related to the symmetry of the stress tensor and the demand for independent external degrees of freedom.

Morley [8], making use of the same analogy mentioned above, has given an equilibrium element, in the sense of Kirchhoff theory, for the plate bending problem. This element is the analog of a compatible two dimensional elasticity element of which there are several to chose from.

The authors are indebted to the referees for bringing [5], [6] and [8] to their attention.

Others who have dealt with equilibrium models are Hessel [9] and Shubinski [10]. Both utilized a nine block assembly of rectangles to form a "building block" element. However, for reasons which are brought out later, these elements have problems, most of which are associated with meeting traction boundary conditions. In addition, these elements are rather poor from a computational point of view.

In order to best illustrate the usage of equilibrium models, it is advantageous to restrict the discussion to the simplest class of problems that still exhibit the general properties of the procedure. In this paper the static generalized plane stress problem with no body forces will be used although it should be noted that the extension to three dimensions is direct.

2. DEVELOPMENT OF THE EQUILIBRIUM STRESS MODEL

First, a comment on notation is in order. Any explicit formulation of a finite element scheme requires many types of super-scripts or subscripts. Tensors, of course, also utilize subscripts and the reader is expected to be able to make all necessary distinctions. Summation convention for repeated indices will be used whether the subscripted quantities are tensor components or not. Finally, it is noted that where tensors are used, Cartesian tensors are implied.

Consider the domain to be divided into E triangular elements. The geometrical approximations necessary at the boundary of the domain are the usual necessary for finite element schemes. Within each element the state of stress will be approximated by a family of M_e simple stress solutions, $\psi_{ij}^{(k)}, (k = 1 \text{ to } M_e)$, i.e., within each element the possible stress field is given by

$$\tau_{ij} = S^{(k)}\psi_{ij}^{(k)} \tag{1}$$

where each $\psi_{ij}^{(k)}$ satisfies:

$$\psi_{ij}^{(k)} \in C^1 \tag{1a}$$

$$\psi_{ij}^{(k)} = \psi_{ji}^{(k)} \tag{1b}$$

$$\psi_{ij,j}^{(k)} = 0. \tag{1c}$$

The comma in the last expression above indicates the partial derivative with respect to x_j. This notation will be used without comment from here on. For simplicity each element is assumed to have the same number and type of trial stress solutions. The parameters, $S^{(k)}$, appearing in equation (1) are referred to as "generalized stresses."

A suitable set of ψ_{ij} satisfying the above for the two-dimensional case may be found by use of the classical Airy stress function. For the models herein this stress function is taken as:

$$\Phi = \frac{E}{\sqrt{E_0}}\left[\left(\frac{x_2^2}{2}\right)\left(\frac{x_1^2}{2}\right)(-x_1x_2)\left(x_1^3\frac{\sqrt{A}}{6}\right)\left(x_1x_2^2\frac{\sqrt{A}}{2}\right)\left(x_1^2x_2\frac{\sqrt{A}}{2}\right)\left(x_2^3\frac{\sqrt{A}}{6}\right)\right]\underline{S} \tag{2}$$

where

E = elastic modulus of the material
E_0 = dimensionless constant of magnitude equal to reference modulus
A = area of element.

This stress function leads to the following seven trial stress solutions:

$$\psi = [\underline{\psi}^{(1)}\underline{\psi}^{(2)}\cdots] = \frac{E}{\sqrt{(AE)_0}}\begin{bmatrix} \sqrt{A} & 0 & 0 & 0 & x_1 & 0 & x_2 \\ 0 & \sqrt{A} & 0 & x_1 & 0 & x_2 & 0 \\ 0 & 0 & \sqrt{A} & 0 & -x_2 & -x_1 & 0 \end{bmatrix} \tag{3}$$

where

$$\underline{\psi}^{(k)} = \begin{Bmatrix} \psi_{11}^{(k)} \\ \psi_{22}^{(k)} \\ \psi_{12}^{(k)} \end{Bmatrix}.$$

Equation (1) now takes the form:

$$\overset{(l)}{\underline{\tau}} = \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{Bmatrix}_{l\text{th element}} = \underline{\psi}\begin{Bmatrix} S^{(1)} \\ \vdots \\ S^{(7)} \end{Bmatrix}_{l\text{th element}} = \underline{\psi}\overset{(l)}{\underline{S}}. \tag{4}$$

Notice that due to the low order of the polynomials involved in Φ that it satisfies the biharmonic equation trivially which means that the resulting strain field will be compatible. This is not required in the formulation, but is desirable and aids in interpretation of displacement information supplied by the solution.

Use is now made of the Theorem of Minimum Complementary Energy. The form of the theorem is the same as given by Sokolnikoff [11]. However, it is necessary to generalize the theorem slightly to include the case where the trial stress solutions are only piecewise continuous (plus other restrictions as set forth below). A statement of the theorem follows:

The complementary energy, π^*, defined below attains a minimal value for the true stress field.

$$\pi^* = \frac{1}{2}\sum_{l=1}^{E}\int_{V^{(l)}} \overset{(l)}{\underline{\tau}}{}^T\underline{E}^{-1}\overset{(l)}{\underline{\tau}}\,\mathrm{d}V - \sum_{l=1}^{E}\int_{\substack{\text{surface of } l\text{th element} \\ \text{where } u_i \text{ is prescribed.}}} T_i\bar{u}_i\,\mathrm{d}A \tag{5}$$

A bar over a quantity indicates that it is prescribed and $\underline{E}^{-1}$ is the matrix of elastic constants of the material. τ_{ij} must be a trial stress field with the properties that

$$\begin{aligned} \tau_{ij} &\in C^1 \\ \tau_{ij} &= \tau_{ji} \\ \tau_{ij,j} &= 0 \end{aligned} \tag{5a}$$

within each of a finite number of subdomains of the body (elements) and must meet the prescribed traction boundary conditions

$$T_i = \tau_{ij} v_j = \overline{T_i} \tag{5b}$$

(v_j are the components of the outward unit normal)

on the exterior boundary segments and T_i is continuous from element to element. (This can be generalized somewhat, see [12].)

Substitution of equation (4) into (5) results in:

$$\pi^* = \frac{1}{2} \sum_{l=1}^{E} \overset{(l)}{\underline{S}}{}^T \overset{(l)}{\underline{f}} \overset{(l)}{\underline{S}} - \sum_{l=1}^{E} \overset{(l)}{\underline{S}}{}^T \overset{(l)}{\underline{s}} \tag{6}$$

where the kpth member of $\overset{(l)}{f}$ is given by

$$\overset{(l)}{f}_{kp} = \int_{V^{(l)}} \underline{\psi}^{(k)T} \underline{E}^{-1} \underline{\psi}^{(p)} \, dV \tag{7}$$

and the kth component of $\overset{(l)}{\underline{s}}$ is given by:

$$\overset{(l)}{s}{}^{(k)} = \int_{\substack{\text{surface of the } l\text{th element where } u_i \text{ is} \\ \text{prescribed.}}} \psi_{ij}^{(k)} v_j \bar{u} \, dS \tag{8}$$

Utilizing equation (3) leads to the form of $\overset{(l)}{\underline{f}}$ for the specific example in question:

$$\overset{(l)}{\underline{f}} = \frac{tE}{E_0} \begin{bmatrix} \alpha A & & & & & & \\ \beta A & \alpha A & & & & & \\ 0 & 0 & \gamma A & & & & \\ \beta x_1^* & \alpha x_1^* & 0 & \alpha\delta_1 & \text{Symmetric} & & \\ \alpha x_1^* & \beta x_1^* & -\gamma x_2^* & \beta\delta_1 & (\alpha\delta_1 + \gamma\delta_2) & & \\ \beta x_2^* & \alpha x_2^* & -\gamma x_1^* & \alpha\delta_{12} & (\beta+\gamma)\delta_{12} & (\alpha\delta_2 + \gamma\delta_1) & \\ \alpha x_2^* & \beta x_2^* & 0 & \beta\delta_{12} & \beta\delta_{12} & \beta\delta_2 & \alpha\delta_2 \end{bmatrix} \tag{9}$$

where: t = thickness of element

$$x_1^* = \frac{1}{\sqrt{A}} \int_{\text{area}} x_1 \, dx_1 \, dx_2 \tag{9a}$$

$$\delta_1 = \frac{1}{A} \int_{\text{area}} (x_1)^2 \, dx_1 \, dx_2 \tag{9b}$$

x_2^* and δ_2 are similar to the above and

$$\delta_{12} = \frac{1}{A}\int_{\text{area}} x_1\, x_2\, dx_1\, dx_2 \tag{9c}$$

α, β, and γ are elastic constants given by

$$\begin{aligned} \alpha &= 1 \\ \beta &= -\sigma \qquad \sigma = \text{Poisson's ratio} \\ \gamma &= 2(1+\sigma) \end{aligned} \tag{9d}$$

for plane stress or by

$$\begin{aligned} \alpha &= 1-\sigma^2 \\ \beta &= -\sigma(1+\sigma) \\ \gamma &= 2(1+\sigma) \end{aligned} \tag{9e}$$

for plane strain.

No undue complexity is introduced if the origin is taken at the center of area of the element so that x_1^* and x_2^* are zero. The axes could be rotated so as to eliminate δ_{12} also. However, this last step introduces almost as much complexity as it eliminates from a computational point of view and will not be done here. Instead, every coordinate system will be aligned with a single "global" system.

Equation (6) may be rewritten as

$$\pi^* = \tfrac{1}{2}\underline{S}^T\underline{f}\underline{S} - \underline{S}^T\underline{s} \tag{10}$$

where the usual finite element notation has been employed, i.e.

$$S = \begin{Bmatrix} \overset{(1)}{\underline{S}} \\ \vdots \\ \overset{(E)}{\underline{S}} \end{Bmatrix} \qquad \text{(similar for } \underline{s}\text{)} \tag{10a}$$

and

$$f = \begin{bmatrix} \overset{(1)}{\underline{f}} & 0 & \cdots \\ 0 & \overset{(2)}{\underline{f}} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{10b}$$

A question arises regarding the components of $\underline{s}$ when no displacements are prescribed on a particular element. For bookkeeping purposes the full $\underline{s}$ is used in the second term on the right of equation (10) and when no displacements are prescribed, then the corresponding components of $\underline{s}$ are given the value of zero. This appears at first to be equivalent to a prescribed boundary displacement of zero but this condition will be "over-ridden" later with a constraint on the force vector corresponding to that boundary. This means that leaving off a force constraint equation is equivalent to prescribing the corresponding

boundary displacement to be zero. The components of the $M \times 1$ array, $\underline{s}$, will be referred to as the "generalized displacements".

As it now stands, the form for the generalized displacement, equation (8), is not in a convenient form for prescribing displacements independently along each boundary. This will be remedied later.

For the special situation when all (interior as well as exterior) displacement boundary conditions are specified, the Theorem of Minimum Complementary Energy can be applied directly since there are no constraints on the components of $\underline{S}$, each being a truly independent parameter. This produces the equation

$$\underline{f}\,\underline{S} = \underline{s} \tag{11}$$

which decomposes immediately to E equations of the form

$$\overset{(l)(l)}{\underline{f}}\,\overset{(l)}{\underline{S}} = \overset{(l)}{\underline{s}}. \tag{11a}$$

The latter equation is usually referred to as the force–displacement relation for the element.

To apply the Theorem of Minimum Complementary Energy in the general case to equation (10) requires some further work since each component of $\underline{S}$ no longer is an independent parameter representing a solution τ_{ij} satisfying the conditions set forth in equation (5a) above.

To develop these constraint equations on $\underline{S}$ it is necessary to introduce another set of parameters $\underline{S}^*$ which describes the state of stress on each boundary independently. Consider a single boundary segment of an element. Since there are a finite number of independent trial stress fields (the $\psi_{ij}^{(k)}$) in the element, only a finite number of independent functions will be needed to describe the components of the stress vector everywhere on this boundary segment. This can be written in equation form as:

$$\mathbf{T} = S_{(k)}^{*}\boldsymbol{\theta}^{(k)}. \tag{12}$$

Here $\mathbf{T}$ and $\boldsymbol{\theta}^{(k)}$ are vectors expressed on some suitable basis such as the usual Cartesian $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and whose components are functions of the position on the surface of the element.

For a single boundary segment the number of stress modes, $\boldsymbol{\theta}^{(k)}$, will in general be less than M_e, the number of trial stress solutions since the $\psi_{ij}^{(k)}$ when evaluated on the boundary segment will no longer be necessarily independent.

The $S_{(k)}^{*}$ for all the boundary segments of the lth element are ordered into a single $M_e^{*} \times 1$ array, $\overset{(l)}{\underline{S}}{}^{*}$. The components of this array must be related to the components of $\overset{(l)}{\underline{S}}{}^{*}$ and in fact must be completely determined by them. Therefore, there must exist a relation of the form:

$$\overset{(l)}{\underline{S}}{}^{*} = \overset{(l)(l)}{\underline{C}}\,\overset{(l)}{\underline{S}} \tag{13}$$

$\underset{\sim}{\overset{(l)}{C}}$ is a $M_e^* \times M_e$ array of numbers determined by the choice of $\psi_{ij}^{(k)}$ and $\theta_i^{(p)}$ and the geometry of the element.

For the element being developed it can be seen from equation (3) that the stress field is essentially one of linearly varying character although not strictly so since the components of the stress tensor are coupled due to the equilibrium requirement. It can be seen then that general nature of the stress vector on a surface is one of linear variation. A convenient choice of mode shapes is illustrated in Fig. 1.

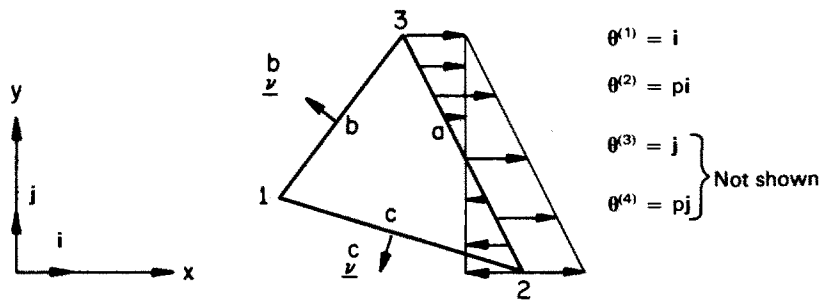


FIG. 1. Arbitrary linearly varying stress element showing boundary stress modes used. Parameter p takes on values from -1 to 1 in moving from node 2 to 3.

Since the character of the stress vector cannot be more complicated than linear, this choice of boundary stress modes is sufficient to describe all possible stress vectors and reflection on equation (3) reveals that the four modes are indeed independent on a *single* boundary i.e., any of the four boundary stress modes may assume an arbitrary value while the remaining three are unchanged.

The vertices or nodes of the element are numbered one through three in a counterclockwise fashion as shown in Fig. 1. The opposite sides are lettered a, b, c respectively. A coordinate system x_1, x_2 is chosen with its origin at the center of area of the triangle and parallel to a "global" system. The coordinates of the ith vertex are designated X_i, Y_i.

To aid in evaluating the stress vector in terms of the element stress field, a parameter p is introduced which gives a parametric representation of the edge a as:

$$\begin{aligned} x_1 &= \tfrac{1}{2}(X_2 + X_3) + \tfrac{1}{2}p(X_3 - X_2) \\ x_2 &= \tfrac{1}{2}(Y_2 + Y_3) + \tfrac{1}{2}p(Y_3 - Y_2) \end{aligned} \tag{14}$$

where p takes on values -1 to 1 to generate the edge a from vertex 2 to vertex 3.

To evaluate the stress vector $\overset{(a)}{\mathbf{T}}$ along the edge a, equations (3), (4), (5b), and (14) are utilized to obtain:

$$\overset{(a)}{\mathbf{T}} = \frac{E}{2L_a\sqrt{(AE_0)}}[\mathbf{i}\ \mathbf{j}]\begin{bmatrix} 1 & p & 0 & 0 \\ 0 & 0 & 1 & p \end{bmatrix}\underset{\sim}{\overset{(a)}{C}}\underline{S} \tag{15}$$

where

$$\underset{\sim}{\overset{(a)}{C}} = \begin{bmatrix} -2\sqrt{A}(Y_2-Y_3) & 0 & 2\sqrt{A}(X_2-X_3) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{A}(X_2-X_3) & -2\sqrt{A}(Y_2-Y_3) & X_2^2-X_3^2 \\ 0 & 0 & 0 & -(X_2-X_3)^2 \end{bmatrix}$$

$$\begin{bmatrix} -2(X_2Y_2-X_3Y_3) & -(X_2^2-X_3^2) & -(Y_2^2-Y_3)^2 \\ 2(X_2-X_3)(Y_2-Y_3) & (X_2-X_3)^2 & (Y_2-Y_3)^2 \\ Y_2^2-Y_3^2 & 2(X_2Y_2-X_3Y_3) & 0 \\ -(Y_2-Y_3)^2 & -2(X_2-X_3)(Y_2-Y_3) & 0 \end{bmatrix}. \tag{16}$$

Utilizing the previously defined boundary stress modes the stress vector on side a is also represented as:

$$\overset{(a)}{\mathbf{T}} = \frac{E}{\sqrt{E_0}}[\mathbf{i}\ \mathbf{j}]\begin{bmatrix} 1 & p & 0 & 0 \\ 0 & 0 & 1 & p \end{bmatrix}\begin{Bmatrix} S_1^* \\ \cdot \\ \cdot \\ \cdot \\ S_4^* \end{Bmatrix}. \tag{17}$$

Comparison of equations (15) and (17) yields:

$$\frac{1}{2L_a\sqrt{A}}\underset{\sim}{\overset{(a)}{C}}\underset{\sim}{S} = \begin{Bmatrix} S_1^* \\ \cdot \\ \cdot \\ \cdot \\ S_4^* \end{Bmatrix}. \tag{18}$$

The same procedure is applied to sides b and c to obtain the desired results:

$$\underset{\sim}{\overset{(l)}{C}}\ \underset{\sim}{\overset{(l)}{S}} = \underset{\sim}{\overset{(l)}{S^*}} \tag{13}$$

where

$$\underset{\sim}{\overset{(l)}{C}} = \begin{bmatrix} \frac{1}{L_a}\underset{\sim}{\overset{(a)}{C}} \\ \hline \frac{1}{L_b}\underset{\sim}{\overset{(b)}{C}} \\ \hline \frac{1}{L_c}\underset{\sim}{\overset{(c)}{C}} \end{bmatrix} \tag{13a}$$

$\underset{\sim}{\overset{(b)}{C}}$ and $\underset{\sim}{\overset{(c)}{C}}$ are obtained from $\underset{\sim}{\overset{(a)}{C}}$ by cyclic permutation of the indices.

The set of equations (13) for each element are ordered into a single set for the structure in the usual fashion.

$$\underset{\sim}{S^*} = \underset{\sim}{C}\underset{\sim}{S}. \tag{19}$$

The $\underset{\sim}{C}$ here is of the same form as the $\underset{\sim}{f}$ of equation (10b).

With the aid of these boundary stress modes, the prescribed traction boundary conditions, equation (5b), can be met provided the loading is restricted to forces representable by the boundary stress modes, i.e., the prescribed stress vector is expanded as:

$$\bar{\mathbf{T}} = R_{(k)}\boldsymbol{\theta}^{(k)}. \tag{20}$$

This restriction is analogous to the one on displacement boundary conditions in the compatible element method.

Equations (20) and (12) are now utilized to produce equations of the form:

$$S^*_{(k)} = R_{(k)}. \tag{21}$$

The requirement for continuity of the stress vector across the imaginary cut between elements is expressed as:

$$\underset{\substack{k\text{th element}\\ \text{common boundary}}}{\mathbf{T}} + \underset{\substack{m\text{th element}\\ \text{common boundary}}}{\mathbf{T}} = \mathbf{0} \tag{22}$$

which results in equations of the form:

$$S^*_{(i)} + S^*_{(j)} = R_{(k)} \quad (R_{(k)} \text{ is zero here}) \tag{23}$$

The S^* components in equation (23) are from the adjoining kth and mth elements as indicated in equation (22). The idea here is trivial but is awkward to express. All of the equations of the form of (21) and (23) are gathered and the components involved are renumbered to produce a set of equations indicated by:

$$\underset{(N\times M^*)}{\underline{A}^*}\ \underset{(M^*\times 1)}{\underline{S}^*} = \underset{N\times 1}{\underline{R}}. \tag{24}$$

$\underline{A}^*$ is made up of only ones and zeros and expresses the connectivity of the element system and is easily generated automatically on a digital computer. For the general situation one includes all of the boundary segments of the elements so that $\underline{S}^*$ is the full $M^* \times 1$ array. Components of $\underline{R}$ (N in number) and their corresponding equations are introduced only where stress boundary conditions exist or across interior "cuts". Introducing equation (19) into (24) gives:

$$\underline{A}^*\underline{C}\underline{S} = \underline{R}$$

or

$$\underset{(N\times M)}{\underline{A}}\ \underset{(M\times 1)}{\underline{S}} = \underset{(N\times 1)}{\underline{R}} \tag{25}$$

with:

$$\underline{A}^*\underline{C} = \underline{A}. \tag{25a}$$

These equations then provide the constraints on $\underline{S}$ so that the prescribed boundary conditions will be met by the trial stress solution.

A more convenient form of generalized displacement will now be introduced. Substitution of equation (12) into the surface integral of equation (5) gives

$$\sum_{l=1}^{E} \underline{S}^{*T}_{(l)}\underline{s}^*_{(l)} = \underline{S}^{*T}\underline{s}^* = \underline{S}^T\underline{C}^T\underline{s}^*$$

where the kth component of $\underline{s}^*$ for the lth element is given by:

$$\overset{(l)}{s^*}_{(k)} = \int \bar{u}_i \theta_i^{(k)} \, dS. \tag{26}$$

The integral extends only over the appropriate boundary segment.

The above replaces the right hand term of equation (6) so that it reads:

$$\pi^* = \tfrac{1}{2}\underline{S}^T \underline{f}\,\underline{S} - \underline{S}^T \underline{C}^T \underline{s}^*. \tag{27}$$

Again all components of $\underline{s}^*$ are carried along for bookkeeping reasons and if displacements are not specified a value of zero is given the corresponding component of $\underline{s}^*$.

The situation is now summarized. The quantity π^* given by equation (27) is to be minimized with respect to the parameters $\underline{S}$ subject to the constraint conditions expressed by:

$$\underline{A}\,\underline{S} = \underline{R}. \tag{25}$$

The array $\underline{f}$, $\underline{s}^*$, $\underline{C}$, and $\underline{R}$ are known and the components of $\underline{S}$ are the desired unknowns.

The problem is now in the form of the classical minimization problem with side conditions and two avenues of solution are immediately available. The first, corresponding to the so-called "force method" in matrix structural analysis, is to utilize the constraint equations, (25), to eliminate N of the unknowns from the problem and then to proceed normally with the minimization of π^*. See the work of Watwood [12] for further discussions of this approach.

The other alternative is the introduction of Lagrangian multipliers into the problem, i.e. to equation (27) is added a term of the form:

$$\underset{1\times N}{\underline{\lambda}^T} (\underline{A}\,\underline{S} - \underline{R})$$

where each component of $\underline{\lambda}$ corresponds to an equation in the set of equations (25).

The minimization of π^* is now carried out in the usual manner. Algebraically this proceeds by applying the necessary condition

$$\frac{\partial \pi^*}{\partial S_{(k)}} = 0$$

giving:

$$\underline{f}\,\underline{S} + \underline{A}^T \underline{\lambda} = \underline{C}^T \underline{s}^*. \tag{28}$$

When equations (25) are added to the above this gives the set:

$$\begin{matrix} M \\ \downarrow \\ \updownarrow \\ N \\ \downarrow \end{matrix} \left[\begin{array}{c:c} \underline{f} & \underline{A}^T \\ \hdashline \underline{A} & \underline{0} \end{array}\right] \left\{\begin{array}{c} \underline{S} \\ \hdashline \underline{\lambda} \end{array}\right\} = \left\{\begin{array}{c} \underline{C}^T \ \underline{s}^* \\ \hdashline \underline{R} \end{array}\right\}. \tag{29}$$

$$\underset{\leftarrow M \rightarrow | \leftarrow N \rightarrow |}{} \quad \underset{(M+N)\times 1}{} \quad \underset{(M+N)\times 1}{}$$

If desired, the set of equations (29) may be solved directly. This procedure corresponds to the method advocated by Klein [13] in matrix structural analysis.

The procedure which leads to the displacement (or stiffness or direct stiffness) method is to look upon equation (29) as two equations and two unknowns and solve as follows:

$$\begin{aligned} \underline{S} &= -\underline{f}^{-1}\underline{A}^T\underline{\lambda}+\underline{f}^{-1}\underline{C}^T\underline{s}^* \\ \underline{A}\underline{S} &= -\underline{A}\underline{f}^{-1}\underline{A}^T\underline{\lambda}+\underline{A}\underline{f}^{-1}\underline{C}^T\underline{s}^* = \underline{R} \\ \underline{\lambda} &= -\underline{K}^{-1}\underline{R}+\underline{K}^{-1}\underline{A}\underline{f}^{-1}\underline{C}^T\underline{s}^* \end{aligned} \tag{30}$$

where

$$\underline{K} = \underline{A}\underline{f}^{-1}\underline{A}^T \tag{30a}$$

giving finally:

$$\underline{S} = \underline{f}^{-1}\underline{A}^T\underline{K}^{-1}\underline{R}-\underline{f}^{-1}\underline{A}^T\underline{K}^{-1}\underline{A}\underline{f}^{-1}\underline{C}^T\underline{s}^*+\underline{f}^{-1}\underline{C}^T\underline{s}^*. \tag{31}$$

The reader, familiar with the work of Argyris [14] will immediately recognize the form of equation (30a) for the stiffness matrix $\underline{K}$.

The above procedure is possible only, of course, if $\underline{f}^{-1}$ and $\underline{K}^{-1}$ exist. Due to the construction of $\underline{f}$, finding $\underline{f}^{-1}$ entails finding $\overset{(l)}{\underline{f}}{}^{-1}$ for every l, i.e. every element. Provided the stress solutions, the $\overset{(k)}{\psi_{ij}}$, are independent functions and that the material is not "incompressible", the resulting $\overset{(l)}{\underline{f}}$ will be nonsingular. For the case of an incompressible material several possibilities exist to overcome the problem, one of which is discussed in [12].

The existence of $\underline{K}^{-1}$ is of more interest since it directly reflects upon the choice of model used. First of all it can be seen from the form of equation (30a) that $\underline{A}$ must be of rank N since the rank of a matrix product is at most equal to the lowest of the ranks of the two matrices involved in the product. Note that among other things, this demands that M be greater than or at most equal to N. This requirement shows up in classical structural analysis in discussions regarding the degree of redundancy $(M-N)$. The fact that $(M-N)$ is equivalent to the classical degree of redundancy becomes evident when one applies the ideas herein to a simple beam element. A negative degree of redundancy in the classical sense means that one has an unstable structure. This situation is quite possible physically for a "pin-jointed" structure. However, for a continuum such a situation does not exist physically provided the body is "supported" so that if N exceeds M one must look to the mathematical model for the problem.

A somewhat more subtle situation occurs when the rows of $\underline{A}$ are not independent. This can be traced directly to the fact that the components of $\underline{R}$ (or $\underline{S}^*$ for a structure composed of a single element) are related by "contraints" aside from overall equilibrium requirements.

The simplest such constraint occurs at a "corner" or "vertex" of an element. At that point only one state of stress can exist and the number of the components of the stress vectors on each "side" of the vertex outnumbers the independent components of the stress tensor since it is symmetric. This situation is, of course, present with the element being developed here. Another constraint which may develop is the situation where two imaginary "cuts" cross. This latter constraint is similar to the above and both owe their origin to the symmetry of the stress tensor. It is eliminated *a priori* in combinations of element assemblies where the first constraint above has been eliminated, but may occur within certain building block assemblies such as a quadrilateral.

Two guidelines are given to overcome the problems with the existence of $\underline{K}^{-1}$ outlined above. These are:

(a) The element must have enough independent stress modes on each boundary segment to react or balance any combination of loads applied on the other boundaries, i.e., the element must be "supportable" from any single boundary segment. This requirement is satisfied by the element being developed but is not for a constant stress element.

(b) The only constraint on the components of $\overset{(l)}{\underline{S}}{}^*$ must be the one of overall equilibrium. This guideline immediately rules out any element which has a vertex in which the components of the stress tensor are continuous in the neighborhood of the "tip" of the vertex. This is overcome by bisecting the vertex with a "cut" across which continuity of the stress vector only is maintained.

The motivation for the first guideline above is that since a single boundary segment is finite in length, it should be possible to "support" a group of elements by fixing a single boundary segment. The motivation for the second can be seen by considering a problem where the body is approximated by a single element. Here one does not want any constraints on the permissible loading other than the overall equilibrium requirement when it is pertinent.

Finally, a convenient criteria for checking possible "building block" elements is that for an arbitrary assembly of the elements with no specified boundary conditions, the following equation should be satisfied:

$$M - N + \Gamma \geq 0. \tag{32}$$

Here Γ is the number of independent overall equilibrium equations of the body, i.e., six for three dimensional problems and three for two dimensional problems. A failure to meet the condition set by equation (32) may be due to failure of either or both of the guidelines set forth previously. But on the other hand, satisfaction of equation (32) does not necessarily imply that both of the guidelines are met. In particular, it is possible to construct an example where equation (32) is always satisfied and yet the element does not meet guidelines (a) and is unsuitable as a primary building block for general situations.

The concept of the "building block" has been introduced as being self-evident. Perhaps some elaboration in connection with the specific example in question is justified on this point. It has been implied in the formulation thus far that planar regions would be approximated with an arbitrary assemblage of triangles (or other simple shape). This is basic to the finite element method. The properties of each such element is desired along with a means of providing the necessary equilibrium continuity requirements. To provide this element, it will be formed of an assemblage of basic elements whose properties have been developed so far, but which must be assembled in a particular manner which avoids the instability problems outlined above. The building block suggested here is shown in Fig. 2.

For this assembly it is seen that M is 21 and N is 24 and equation (32) is satisfied. Since there are no redundant equilibrium conditions in the assembly, it is not necessary to reduce out the interior cuts. That is, one may approximate the region under consideration with a group of triangles such as 1–2–3 in Fig. 2 in an arbitrary manner and then let the computer introduce the additional cuts, such as 01, 02, and 03 in Fig. 2. The solution may then proceed as summarized by equation (31) without any reformulation of the $\overset{(l)}{\underline{f}}$ and $\overset{(l)}{\underline{S}}$ involved.

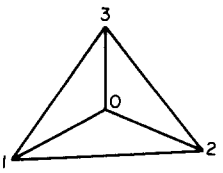


FIG. 2. Building block assembly for linearly varying stress field element.

However, such a procedure increases the size of M and N for the complete assembly quite rapidly and one is led to suspect that due to the geometry introduced in the subdividing procedure that it would be worthwhile to reduce out the interior degrees of freedom.

To accomplish the required reduction, the equilibrium equations for the assembly are written (corresponding to equations (25)). In so doing the interior force degrees of freedom are placed at the top of $\underline{R}$, i.e. the equations are of the form

$$\begin{bmatrix} \underline{A}_u \\ \hline \underline{A}_l \end{bmatrix} S = \begin{Bmatrix} \underline{0} \\ \hline \underline{R}_e \end{Bmatrix} \tag{33}$$

where the components of $\underline{R}_e$ are the exterior force degrees of freedom and $\underline{A}$ is partitioned as shown. $\underline{A}_u$ is, of course, the matrix of coefficients of the equilibrium equations obtained by demanding continuity of the stress vector across each interior cut. In this case $\underline{A}_u$ is of rank twelve and there exists a nonsingular matrix $\underline{Q}$ (see example Beaumont and Ball [15]) such that:

$$\underline{A}_u \underline{Q} = \left[\underline{I} \;\vdots\; \underline{0} \right] \tag{34}$$

a transformation is then introduced of the form:

$$\underline{S} = \underline{Q}\underline{S}' = \left[\underline{Q}_0 \;\vdots\; \underline{Q}_1 \right] \begin{Bmatrix} \underline{S}'_u \\ \underline{S}'_l \end{Bmatrix} \tag{35}$$

$\underline{Q}$ is partitioned between the 12th and 13th columns and $\underline{S}'$ is partitioned accordingly. Equation (35) is then introduced into the upper equations of (33) with the result:

$$\begin{gathered} \underline{A}_u \quad \underline{Q}\,\underline{S}' = \underline{0} \\ \left[\underline{I} \;\vdots\; \underline{0} \right] \begin{Bmatrix} \underline{S}'_u \\ \underline{S}'_l \end{Bmatrix} = \underline{0}. \\ \underline{S}'_u = \underline{0}. \end{gathered} \tag{36}$$

Hence, equation (35) takes the form:

$$\underset{21\times 1}{\underline{S}} = \underset{21\times 9}{\underline{Q}_1} \; \underset{9\times 1}{\underline{S}'_l} \tag{37}$$

and the lower equations of (33) produce:

$$\underline{A}_l \underline{Q}_1 \underline{S}'_l = \underline{R}_e \tag{38}$$

The matrix product of $\underline{A}_I\underline{Q}_1$ is now considered as the matrix $\overset{(l)}{\underline{C}}$ for the building block element and the matrix $\underline{S}'_I$ becomes the generalized stresses. After the primary unknowns, the $\underline{S}'_I$, are determined, the stress distribution is found from use of equations (37) and three sets of equations of the form of (4).

It now remains to develop the $\overset{(l)}{\underline{f}}$ for this building block element. This is determined directly by substitution of equation (37) into equation (6) to give

$$\overset{(l)}{\underline{f}} = \underline{Q}_1^T \underline{f} \underline{Q}_1 \tag{39}$$

recalling that $\underline{f}$ as used here is made up of individual seven by seven flexibility matrices of the three elements making up the building block.

It may be noted in passing that $\underline{Q}$ is not unique and, in fact, it is not necessary to reduce $\underline{A}_u$ completely to the form shown in equation (34). All that is required is that a non-singular square matrix appear in place of $\underline{I}$. Also it is seen that only the last nine columns of $\underline{Q}$ are utilized. These observations may be capitalized upon in the actual reduction scheme utilized in order to conserve computer time. The details of the reduction scheme are not given here as it is basically a Gaussian reduction done with columns rather than rows as is the usual scheme. For a more complete description see [12].

Finally the question of displacement field information will be considered. First of all, provided that the trial stress solutions are compatible, then a displacement field within each element can be found by integration. This displacement field is found, as would be expected, only to within rigid body displacements. Also it should be pointed out that there is no reason why compatible stress fields would not always be used since it adds no complexity to the solution. In fact, for the example being developed here, the stress fields are compatible and not by design. The compatibility is a result of the simple form of the stress fields used.

Rigid body information is supplied only in a "mean" or integrated sense by the coefficients of the Lagrangian multiplier vector $\underline{\lambda}$. This is obvious to the experienced elasticity analyst (e.g. see Pearson [16], pp. 146–7) and therefore the proof will not be given here. The result is, in effect, Castigliano's theorem on displacements for a somewhat more complicated situation than normal. The result follows:

$$\lambda_{(k)} = -\int_{\text{boundary segment corresponding } R_{(k)}} \theta_i^{(k)} u_i \, dS \tag{40}$$

The subscript k on the λ corresponds to the kth equation of equations (25) and therefore corresponds to the kth component of $\underline{R}$. Equation (40) may not appear too helpful at first and will not be unless the boundary stress modes are defined properly. For example, if $\theta_1^{(k)} = 1$ and $\theta_2^{(k)} = 0$, then equation (40) gives the "average" displacement in the x_1 direction. Such simplicity in the stress modes is helpful in extracting information from the solution and was considered in the choice of modes used for the example herein.

An equilibrium element has now been given which is free of peculiar properties and which may be used in a manner consistent with the usual philosophy for finite elements. Needless to say, there are other possibilities. A quadrilateral possibility was suggested by Fraeijs de Veubeke [4] which can be made to work. Fraeijs de Veubeke did not give the details necessary to utilize this element but they may be found in [12]. A suitable assembly made up of constant stress basic elements is also given in [12].

3. NUMERICAL RESULTS

The final test for any proposed numerical procedure is to implement it and actually work some problems for which results are known. This was done for the linearly varying stress field element as developed thus far.

The first problem given here is the cantilever of Fig. 3. The boundary condition is shown in a "fixed" condition. In reality this only infers that (or forces) the generalized displacements of the form of equation (40) to be zero; i.e., the integral of the displacements, both in the x and y directions, "weighted" with a constant and a linearly varying function, are zero. In the first solutions the "fixity" on the linearly varying component of the y-displacement was relaxed. For this situation the "axial" and "moment" loads gave the exact results for any of the element assemblies in Fig. 3 which was to be expected since the trial solutions contain the exact stress solution. The same situation occurs if Poisson's ratio is taken to be zero even if the above boundary condition is not relaxed.

Another check on the method was obtained from the shear loading shown in Fig. 3.

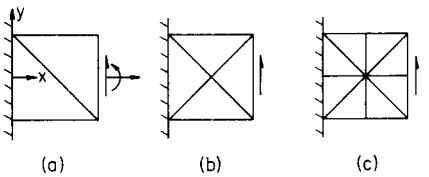


FIG. 3. Various cantilever element representations.

As shown by Synge [1], an "equilibrium" solution should give an upper bound on strain energy if stress boundary conditions are imposed (this includes zero displacement boundary conditions as well). Since the theory involved is classical linear elasticity the strain energy stored is equal to one half of the work that would be done by the applied forces acting through the final displacements from the unstressed state (Clapeyron's Theorem). This says that:

$$\xi_\tau = \frac{1}{2}\int T_i u_i \, ds$$

For the example in question T_2 is constant so that the above is:

$$\xi_\tau = \frac{1}{2} T_2 \int_{\text{end boundary segment.}} u_2 \, ds$$

Therefore since the strain energy here is known to be an upper bound and T_2 is known, then the remaining integral (which is one of the $\underline{\lambda}$ components) must be an upper bound as compared to the integral of the true displacements.

A table of the component of $\underline{\lambda}$ associated with the x_2 deflection of the end of the beam is shown in Table 1 as it varies with an increasing number of elements. The compatible solution indicated, which meets the prescribed displacements exactly and is therefore an opposite bound, is obtained by combining the usual beam theory with a state of constant shear strain and hence stress. Various other solutions exist to this problem with different assumptions regarding the fixed boundary condition and the interior shear stress distribution. However, due to the aspect ratio involved, it is difficult to provide a meaningful

comparison with these "beam type" solutions. The results are seen to be good with the equilibrium model solutions apparently converging rapidly.

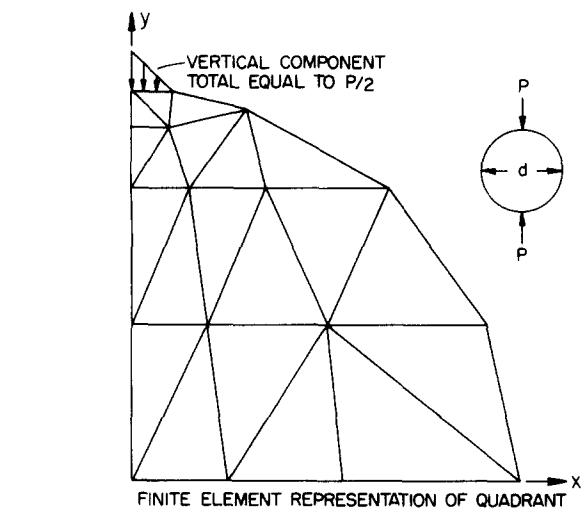
TABLE 1. END DEFLECTION OF A CANTILEVER

Element assemblies are as shown in Fig. 3. Poisson's ratio is zero and beam has unit thickness

Solution	Value of $\delta E/P$
Compatible	6·0
Fig. 3(c)	6·60
Fig. 3(b)	6·62
Fig. 3(a)	11·7

δ = end deflection, P = total end load, E = Young's modulus.

The results of one other problem is given. It is the classical problem of the disc loaded with the concentrated loads as shown in the sketch at the side of Fig. 4. The element breakdown is shown in Fig. 4 and the normal stress distribution is indicated on the bottom of the figure along with the exact results as obtained from Timoshenko [17].



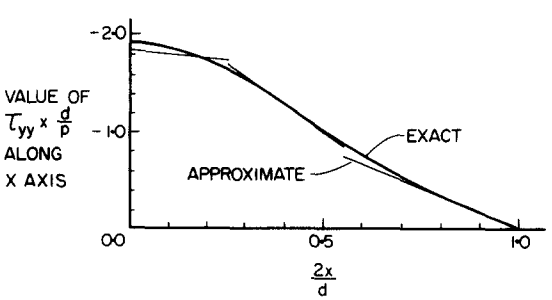


FIG. 4. Disc loaded with two diametric concentrated loads.

A significant check made here was to do this same problem twice but with different elastic constants. Poisson's ratio was taken both at zero and 0·3 and the stress results were the same to at least five significant figures, whereas the displacements differed. This is, of course, the expected result.

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(*Received* 28 *August* 1967; *revised* 14 *March* 1968)

Абстракт—Двойственная формулировка обычной моделя сходного элемента очень пригодна для ирезуемомрения ограничений в точном решении. Элемент, поле напряжения которого удовлетворает дифференциальным уравнéниям равновесия, и связанный с сопряженными элементами таким способом, чтобы обеспечить непериврность иквектора напряжения параллельно всем краям представляет требуемую двойственность.

Определяется пригодная модель равновесия, с целью использования ее на практике для схемы конечного элемента. Указываются направления для формулировки других элеменуов и описываются причины трудоемкости, которые были причиной хлопот предыдущих исследователей.

Даются численные расчеты, которые служат для иллюстрации и доказательства представленных идей.